GRAPH THEOREMS FOR MANIFOLDS†

BY

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ABSTRACT

Two basic theorems about the graphs of convex polytopes are that the graph of a d-polytope is d-connected and that it contains a refinement of the complete graph on $d+1$ vertices. We obtain generalizations of these theorems, and others, for manifolds. We also supply some details for a proof of the lower bound inequality for manifolds.

1. Introduction

The advent of linear programming has brought about a renewal of interest in the combinatorial structure of convex polytopes. In the past ten years many new theorems have been proved for these polytopes. It is interesting that many of them are also true for more general structures such as triangulated manifolds and pseudo-manifolds.

In this paper we shall generalize the following theorems to manifolds and pseudo-manifolds.

(i) The graph of a d-polytope is d-connected [3, p. 213].

(ii) Each vertex of the graph of a d-polytope is contained in a refinement of C_{d+1}, the complete graph on $d+1$ vertices [3, p. 200].

(iii) (Klee [4].) If a set of n vertices separates the graph of a d-polytope, then the number of components is 1 (if $n \leqq d-1$), is 2 (if $n = d$), and is less than or equal to the maximum number of facets of any d-polytope with n vertices for $n \geqq d+1$.

As an application of one of our theorems, we shall prove the Lower Bound theorem for manifolds which states:

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If M is a d-manifold with v vertices, then M has at least $dv - (d+2)(d-1)$ facets.

This theorem was proved for polytopes in [1], where we mention that it can be proved for manifolds although no proof is given.

2. Definitions

A *d-cell complex* is a collection $\mathscr{C}$ of closed k-cells (called *faces* of $\mathscr{C}$) where $-1 \leqq k \leqq d$ such that

(i) the boundary of each k-cell is a $(k-1)$ cell complex.

(ii) the intersection of any two faces $\mathscr{F}_1$ and $\mathscr{F}_2$ is a face of both $\mathscr{F}_1$ and $\mathscr{F}_2$ (possibly the empty face).

(iii) each face of $\mathscr{C}$ is a face of a d-cell of $\mathscr{C}$.

A *simplicial d-cell complex* is one in which the facial structure of each d-cell is isomorphic to the facial structure of the d-simplex.

A *d-manifold* is a compact metric space that is locally homeomorphic to the d-cell. A *triangulated d-manifold* is one that is the union of the faces of some simplicial d-cell complex.

A *d-pseudo manifold* $\mathscr{M}$, is a simplicial d-complex that satisfies the following conditions.

(i) Each $(d-1)$ face belongs to exactly two d-faces.

(ii) $\mathscr{M}$ is strongly connected, that is, given any two d-simplices S_1 and S_n in $\mathscr{M}$ there is a sequence $S_1, S_2, \cdots, S_n$ of simplices such that S_i and S_{i-1} intersect on a $(d-1)$ simplex. (Such a sequence of facets will be called a *strong chain.*)

Every triangulated manifold is a pseudo-manifold but the converse is not true. A *cellular decomposition* of a d-manifold is a d-manifold that is the union of the faces of a d-cell complex.

A *vertex* of a cell complex is a face of dimension 0, an *edge* is a 1-dimensional face. A *facet* of a d-cell complex is a d-face and a *subfacet* is a $(d-1)$-face. The *graph* of a d-cell complex is the linear graph formed by its vertices and edges.

The *body* of a cell complex $\mathscr{C}$, denoted by $|\mathscr{C}|$ is the union of its faces of $\mathscr{C}$. A cell complex $\mathscr{C}$ is a *refinement* of a cell complex $\mathscr{C}'$ if there is a homeomorphism of $|\mathscr{C}'|$ onto $|\mathscr{C}|$ such that the image of a face of $\mathscr{C}'$ is the union of faces in $\mathscr{C}$.

The *k-skeleton* of a cell complex $\mathscr{C}$, denoted by $\mathrm{skel}_k(\mathscr{C})$, is the cell complex of all faces of $\mathscr{C}$ of dimension at most k. The *boundary complex* of a d-polytope P is $\mathrm{skel}_{d-1}(P)$.

Let $\mathscr{F}$ be a face of a cell complex $\mathscr{M}$. We define $\mathrm{ast}(\mathscr{F},\mathscr{M})$, the *antistar* of

$\mathscr{F}$ in $\mathscr{M}$, to be the collection of all faces of $\mathscr{M}$ that do not contain $\mathscr{F}$ as a face. We define star($\mathscr{F}$,$\mathscr{M}$), the *star* of $\mathscr{F}$ in $\mathscr{M}$, to be the collection of all faces of $\mathscr{M}$ containing $\mathscr{F}$ together with all faces that belong to faces containing $\mathscr{F}$. We define link($\mathscr{F}$, $\mathscr{M}$), the *linked complex* of $\mathscr{F}$ in $\mathscr{M}$, to be star($\mathscr{F}$, $\mathscr{M}$) $\cap$ ast($\mathscr{F}$,$\mathscr{M}$).

A graph G is said to be *n-connected* provided it has at least $n + 1$ vertices and cannot be separated by removing fewer than n vertices. We shall use the theorem of Whitney [7] that a graph is n-connected if and only if any two vertices can be joined by n independent paths (that is, paths that meet only at their endpoints).

3. Graph manifolds

In order to prove our theorems for manifolds we shall construct what we call graph manifolds, which can be regarded as a combinatorial generalization of pseudo-manifolds.

A (-1)-*graph manifold* is a graph consisting of one vertex. *A* 0-*graph manifold* is a graph consisting of a single edge and its two vertices. A 1-*graph manifold* is a graph consisting of a simple circuit of at least three edges, its edges and its vertices.

Inductively, an *n-graph manifold*, $\mathscr{M}$, hereafter abbreviated *n-gm*, is a collection, $\mathscr{C}$, of *k-gm*'s ($0 \leqq k \leqq n - 1$), called *faces* of $\mathscr{M}$, such that the following conditions hold.

(i) If $\mathscr{F} \in \mathscr{C}$ then every face of $\mathscr{F}$ is in $\mathscr{C}$.

(ii) The non-empty intersection of any two faces $\mathscr{F}$ and $\mathscr{F}'$ of $\mathscr{M}$ is a face of both $\mathscr{F}$ and $\mathscr{F}'$.

(iii) $\mathscr{M}$ is strongy connected.

(iv) Each face of $\mathscr{M}$ belongs to an $(n - 1)$-face of $\mathscr{M}$.

(v) Each $(n - 2)$-face of $\mathscr{M}$ is in exactly two $(n - 1)$-faces of $\mathscr{M}$.

If (iii) is not satisfied, we call $\mathscr{M}$ a *pseudo-graph manifold*. The *graph of* $\mathscr{M}$ is the graph consisting of vertices and edges of $\mathscr{M}$, and will be denoted by $\mathscr{G}(\mathscr{M})$. The terms vertex, edge, facet and subfacet, star, antistar and link will be used in the same way as for cell complexes. Note that a facet of an *n-gm* is an $(n - 1)$-face, not an n-face. The number n will be called the *dimension* of $\mathscr{M}$.

If M is a triangulated manifold, pseudo-manifold. or cellular decomposition of a manifold, then M can be associated with an *n-gm* $\mathscr{M}$ of the same dimension, such that $\mathscr{M}$ and M are *isomorphic*, that is, there is a 1-1 function taking faces of M onto faces of $\mathscr{M}$ such that dimension and incidences are preserved. In any of the structures in which we work, we shall say that two faces are *incident* provided one face is a face of the other.

LEMMA 1. *Let* $\mathcal{F}_1, \cdots, \mathcal{F}_m$ *be a sequence of distinct facets of an n-gm* $\mathcal{M}$ *such that* $\mathcal{F}_i \cap \mathcal{F}_{i+1}$, $1 \leqq i \leqq m-1$, *is a subfacet of* $\mathcal{M}$ *and such that each* $\mathcal{F}_i$ *contains an* $(n-3)$*-face* F. *Then there exists a sequence of distinct facets* $\mathcal{F}_1, \cdots, \mathcal{F}_m, \mathcal{F}_{m+1}, \cdots, \mathcal{F}_l = \mathcal{F}_1$ *such that* $\mathcal{F}_i \cap \mathcal{F}_{i+1}$ *is a subfacet of* $\mathcal{M}$ *for* $1 \leqq i \leqq l-1$, *and each* $\mathcal{F}_i$ *contains* F *(such a sequence will be called a strong cycle).*

PROOF. Let $\mathcal{F}_{m-1} \cap \mathcal{F}_m = \mathcal{S}_m$. On $\mathcal{F}_m$ the face F belongs to exactly two subfacets of $\mathcal{M}$. Let $\mathcal{S}_{m+1}$ be the other subfacet of $\mathcal{F}_m$ containing F. Let $\mathcal{F}_{m+1}$ be the facet that shares $\mathcal{S}_m$ with $\mathcal{F}_m$. Continuing in this way, we can construct facets $\mathcal{F}_{m+2}, \mathcal{F}_{m+3}, \cdots$, until we reach a facet $\mathcal{F}_{m+j}$ that has appeared earlier in the sequence of $\mathcal{F}_i$. If $\mathcal{F}_{m+j} = \mathcal{F}_1$, we are done. Suppose $\mathcal{F}_{m+j} = \mathcal{F}_i$ for $i \neq 1$. Let $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$ be the two subfacets on $\mathcal{F}_i$ that contain F. The facet $\mathcal{F}_{m+j-1}$ meets $\mathcal{F}_i$ on one of these two subfacets, but either $\mathcal{F}_{i+1}$ or $\mathcal{F}_{i-1}$ also meets $\mathcal{F}_i$ on this subfacet which contradicts condition (v) for *n-gm*'s, and the lemma is proved.

LEMMA 2. *The antistar of any vertex of an n-gm is strongly connected.*

PROOF. Our proof is by induction on n. It is clear that the theorem is true for $n \leqq 1$. Suppose v is a vertex of an *n-gm* $\mathcal{M}$, $n > 1$. Let $\mathcal{F}_1$ and $\mathcal{F}_k$ be two facets in ast(v) and let $\mathcal{C} = \{\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_k\}$ be a strong chain connecting them. If the chain misses v, we are done. Suppose some facet in the chain contains v. Let $\mathcal{F}_i$ be the first such facet and suppose that $\mathcal{F}_i, \mathcal{F}_{i+1}, \cdots, \mathcal{F}_j$ all meet v and that $\mathcal{F}_{j+1}$ does not. Let $\mathcal{S}_i = \mathcal{F}_{i-1} \cap \mathcal{F}_i$ and $\mathcal{S}_{j+1} = \mathcal{F}_j \cap \mathcal{F}_{j+1}$. By induction ast$(v, \mathcal{F}_l)$ for $i \leqq l \leqq j$ is strongly connected. Using this fact, we can get a strong chain $\mathcal{S}_1, \cdots, \mathcal{S}_m$ of $(n-1)$-faces in link$(v, \mathcal{M})$ from $\mathcal{S}_j$ to $\mathcal{S}_j$. For each pair of consecutive subfacets $\mathcal{S}_h$ and $\mathcal{S}_{h+1}$ let $\mathcal{F}_h$ and $\mathcal{F}_{h+1}$ be facets of $\mathcal{M}$ containing $\mathcal{S}_h$ and $\mathcal{S}_{h+1}$ respectively, and not meeting v. The facets $\mathcal{F}_h$ and $\mathcal{F}_{h+1}$ together with either one or two facets of the sequence $\mathcal{F}_i, \cdots, \mathcal{F}_j$ forms a strong sequence of facets containing an $(n-3)$-face of $\mathcal{M}$. By Lemma 1, they belong to a strong cycle. The portion of this strong cycle that misses v will be a strong chain connecting $\mathcal{S}_h$ and $\mathcal{S}_{h+1}$. In this way, we get a strong chain from $\mathcal{S}_i$ to $\mathcal{S}_{j+1}$. Thus we get a strong chain from $\mathcal{F}_{i-1}$ to $\mathcal{F}_{j+1}$. By doing the same thing whenever we encounter facets of $\mathcal{C}$ meeting v, we construct a strong chain missing v.

LEMMA 3. *An n-gm has at least* $n+2$ *vertices.*

PROOF. The proof is an easy induction on n and is left to the reader.

THEOREM 4. *The graph* $\mathscr{G}(\mathscr{M})$ *of an n-gm* $\mathscr{M}$ *is* $(n+1)$*-connected.*

PROOF. By Lemma 3, $\mathscr{M}$ has at least $n+2$ vertices. Suppose V is a set of vertices separating $\mathscr{G}(\mathscr{M})$. Let $v \in V$, and let $\mathscr{F}_1$ and $\mathscr{F}_k$ be facets meeting different components of the separated graph and missing v. Let $\mathscr{C} = \{\mathscr{F}_1, \cdots \mathscr{F}_k\}$ be a strong chain joining $\mathscr{F}_1$ and $\mathscr{F}_k$ in $\text{ast}(v, \mathscr{M})$. The set V must separate the graph of $\mathscr{C}$. If some facet of $\mathscr{C}$ is separated by V, then by induction V has at least n vertices in that facet and thus V has at least $n+1$ vertices. If no facet of $\mathscr{C}$ is separated by V, then some subfacet $\mathscr{S}$ belonging to two consecutive facets of $\mathscr{C}$ will have all of its vertices in V. By Lemma 3, $\mathscr{S}$ has at least n vertices. Thus v has at least $n+1$ vertices.

The theorem is clearly true for $n \leqq 1$, thus the induction can be started.

COROLLARY 5. *The graph of an n-manifold or an n-pseudo manifold is* $(n+1)$*-connected.*

COROLLARY 6. *The graph of an n-polytope is n-connected.*

THEOREM 7. *If* $\mathscr{M}$ *is an n-gm then each vertex belongs to a refinement of* C_{n+2}, *the complete graph on* $n+2$ *vertices, in* $\mathscr{G}(\mathscr{M})$.

PROOF. Let v be a vertex of $\mathscr{M}$. We form a pseudo $(n-1)$-*gm* $\mathscr{M}'$ as follows.

(i) The vertices of $\mathscr{M}'$ correspond to edges of $\mathscr{M}$ that meet v.

(ii) Two vertices are joined by an edge, if and only if the corresponding edges of $\mathscr{M}$ lie on a 1-face of $\mathscr{M}$.

(iii) Inductively, a collection of k-faces of $\mathscr{M}'$ determines a $(k+1)$-face, if and only if the corresponding $(k+1)$-faces of $\mathscr{M}$ all contain v and are the $(k+1)$-faces of $\mathscr{M}$ lying on some $(k+2)$-faces meeting v.

This pseudo *gm* is called the *vertex figure* of v, and will be denoted by $\mathscr{V}(v)$.

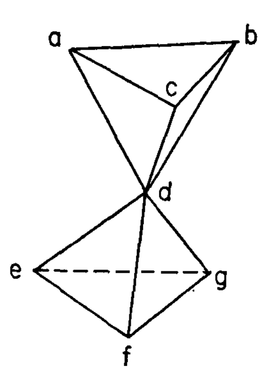

Fig. 1. Two tetrahedra meeting at a common vertex.

We illustrate this with the case where $\mathcal{M}$ is the boundary of two tetrahedra meeting at a common vertex (Fig. 1). The vertices of $\mathcal{V}(v)$ are a, b, c, e, f, and g. The edges are ac, cb, ba, ef, fg, and eg. The maximal strongly-connected components of $\mathcal{V}(v)$ are the circuits abc and efg.

Since $\mathcal{V}(v)$ is clearly a pseudo n-gm, the maximal strongly-connected subcomplexes of $\mathcal{V}(v)$ are n-gm's. Let $\mathcal{M}_1$ be one of these subcomplexes. By induction $\mathcal{G}(\mathrm{M}_1)$ contains a refinement of C_{n+1}, We shall show that corresponding to the refinement of C_{n+1} in $\mathcal{G}(\mathrm{M}_1)$, there is a refinement of $\mathcal{C}_{n+1}$ in $\mathcal{G}(\mathrm{link}(v))$.

Let e be an edge of $\mathcal{M}_1$ and let $\mathcal{F}$ be the 2-face of $\mathcal{M}$ corresponding to e. Let e_1 and e_2 be the two edges of $\mathcal{M}$ corresponding to the endpoints of e and let v_1 and v_2 be the endpoints of e_1 and e_2 in link(v). In $\mathcal{F}$ there is a path from e_1 to e_2 that misses v, thus this path lies in link(v). For each edge of the refinement of C_{n+1} in $\mathcal{G}(\mathrm{M}_1)$ there is a corresponding path in link(v) and these paths form a refinement $\mathcal{G}$ of $\mathcal{C}_{n+1}$ in link(v). Adding the edges from v to the n-valent vertices of $\mathcal{G}$, gives us a refinement of $\mathcal{C}_{n+2}$.

4. The degree of separability

The nth *degree* of *total separability* of a graph $\mathcal{G}$, denoted by $s_n(\mathcal{G})$, is defined to be the maximum number of components obtained by removing n vertices from $\mathcal{G}$. Klee's theorem states essentially that if $\mathcal{G}$ is the graph of a d-polytope then $s_n(\mathcal{G})$ is at most the maximum number of facets of any d-polytope with n vertices. Generalizations of Klee's theorem do not seem to be quite as nice. The first two theorems, as we have seen, are purely combinatorial. When we look at the degree of separability of the graph of a manifold M, topology now becomes important. That is, the degree of separability not only depends on the dimension of M, which we could treat in a purely combinatorial way, but also on the topology of M.

For example, the maximum number of components that the graph of a cellular decomposition of a 2-sphere can be divided into by removing 7 vertices is 10 (this follows from Klee's theorem), yet if we take a triangulation of the torus with 7 vertices and make a stellar subdivision of each 2-face then we have the graph of a 2-manifold that can be divided into 14 components by removing 7 vertices.

It would be tempting to conjecture that the number of components of a graph of a d-manifold separated by n vertices is at most the maximum number of d-cells in any cellular decomposition of that manifold, with n vertices. Unfortunately, there may be many values for n for which there is no cellular decomposition of

$\mathscr{M}$ with n vertices. However if $\mathscr{M}$ has a cellular decomposition with n vertices then it has one for any number of vertices greater than n, thus the conjecture may be true for large enough n. We shall prove some weakened forms of this conjecture, some of which we are able to prove only for 2- and 3-dimensional manifolds.

If $\mathscr{C}'$ is a cellular decomposition of a d-manifold M, and $\mathscr{C}$ is a $(d-1)$-cell subcomplex of $\mathscr{C}'$, we define $C(\mathscr{C}, M)$ to be the number of strongly-connected components of $M \sim \mathscr{C}$. We define $C(k, M)$ to be the maximum of $C(\mathscr{C}, M)$ taken over all $(d-1)$-subcomplexes with k vertices, of all cellular decompositions of M. By a *d-face* of $\mathscr{C}$ we shall mean a strongly-connected component of $M \sim \mathscr{C}$.

THEOREM 8. *Let $\mathscr{G}$ be the graph of a cellular decomposition $\mathscr{C}$, of a 2-manifold M. If a set V of v vertices separates $\mathscr{G}$ then the number of components of the separation is at most $C(v, M)$.*

PROOF. Let $\mathscr{G}_1$ be the subgraph of $\mathscr{G}$ determined by V. If each component of the separation belongs to a different 2-face of $\mathscr{G}_1$ then we are done, so we may assume that at least two components belong to the same 2-face, F', of $\mathscr{G}_1$. If this is the case then some facet F of $\mathscr{C}$ contains edges of two different components in F'. This facet must have vertices on F' for otherwise the two different components would be connected by a path along the facet and, in fact, for this reason the facet F must have two vertices on F' that are not consecutive on F. These two vertices are not joined in $\mathscr{G}$, for if they were, then any facet of $\mathscr{C}$ containing that edge would meet F in a way that is not allowed in cellular decompositions of M. We now add to $\mathscr{G}'$ an edge joining these two vertices, producing a graph $\mathscr{G}_2$. We continue until we have a graph $\mathscr{G}_n$ for which each component lies in a separate 2-face. It is now clear that the conclusion of the theorem is true.

THEOREM 9. *Let M be a triangulated n-pseudo manifold with graph $\mathscr{G}$ and suppose $\mathscr{G}$ is separated by a set V of v vertices. Then the number of components of the separation is at most $C(v, M)$.*

PROOF. Let $\mathscr{C}$ be the $(n-1)$-complex consisting of faces of M whose vertices are in V. This is an $(n-1)$-complex, for if its dimension were less than $n-1$ then we could separate M by removing a set of dimension less than $n-1$ contradicting the fact that M is strongly connected. Suppose some n-face F' of $\mathscr{C}$ contains more than one component of the separated graph. Let two of the components be K_1 and K_2 and let $\{\mathscr{F}_1, \cdots, \mathscr{F}_k\}$ be a strong chain of facets joining them in F'. Let $\mathscr{F}_i$ be the last facet of the chain meeting K_1. The facet $\mathscr{F}_i$ contains only vertices of K_1 and V, for if some other component K_2 met $\mathscr{F}_i$, then an edge of $\mathscr{F}_i$ would join these components in F'. Since F_{i+1} misses K_1, we see that $\mathscr{F}_i \cap \mathscr{F}_{i+1}$ has only vertices of V, in other words, $\mathscr{F}_i \cap \mathscr{F}_{i+1}$ is a face of $\mathscr{C}$. This contradicts the fact that $\{\mathscr{F}_1 \cdots, \mathscr{F}_k\}$ lies in F'.

LEMMA 10. *Any cellular decomposition $\mathscr{C}$ of a 3-sphere can be triangulated without introducing any new vertices.*

PROOF. Let v be a vertex of $\mathscr{C}$. In $\mathscr{C}$ we replace star(v) by the set of 3-cells of the form $v \vee \mathscr{F}$ (the join of v and $\mathscr{F}$) where $\mathscr{F}$ is a face of link(v). We do this to vertices of $\mathscr{C}$ until each facet is pyramidal at each of its vertices. (A facet F is said to be *pyramidal at v* if it is of the form $v \vee \mathscr{F}$ for some face $\mathscr{F}$ of F). An easy proof shows that if a facet is pyramidal at each vertex, it is a simplex.

COROLLARY 11. *The maximum number of facets of any cellular decomposition of a 3-sphere with a given number of vertices is at most the maximum number of facets of any triangulation of the 3-sphere with the same number of vertices.*

THEOREM 12. *If $\mathscr{C}$ is a cellular decomposition of a 3-manifold M and the graph of $\mathscr{C}$ is separated by a set V of v vertices then the number of components of the separation is less than or equal to the maximum number of facets of any cellular decomposition of the 3-sphere with v vertices.*

PROOF. It follows from a theorem of Moise [5] that link(x) is a 2-sphere for any vertex x in $\mathscr{C}$. By a theorem of Steinitz [6] any such 2-sphere is isomorphic to the boundary complex of a 3-polytope. If $x \in V$ then by Klee's theorem the maximum number of components of the separated graph meeting link(x) is less than or equal to the maximum number of facets of any 3-polytope with $v-1$ vertices. Thus the maximum number of such components is $2(v-1)-4 = 2v-6$ (see [3, Ch. 10]). If we sum up the number of components meeting the linked complexes of the vertices in V the sum is at most $v(2v-6)$. Since the graph of $\mathscr{C}$ is 4-connected, each component lies in the linked complex of at least 4 vertices. Thus the total number of components is at most $(v^2-3v)/2$. This, however, is the maximum number of facets of any triangulated 3-sphere with v vertices [3, Ch. 10]. From Corollary 11, we see that this is the maximum number of facets of any cellular decomposition of a 3-sphere with v vertices.

5. The lower bound theorem for simple n-gm's

In this section we supply the details needed to prove the Lower Bound theorem for triangulated d-manifolds. The reader should read [1] in order to understand this section, as we do not wish to repeat the original proof of the Lower Bound theorem here.

In [1] we used a theorem of Sallee which we shall generalize here. We define a *strong n-gm cell complex* $\mathscr{C}$ to be a collection of *n-gm*'s such that given any two *n-gm*'s in $\mathscr{C}$ there is a strong chain of *n-gm*'s joining them.

THEOREM 13. *The graph of a strong n-gm complex* $\mathscr{C}$ *is* $(n+1)$*-connected.*

PROOF. $\mathscr{C}$ has at least $n+2$ vertices by Lemma 3. Suppose a set V of vertices separates $\mathscr{G}(\mathscr{C})$. We choose a strong chain $C = \{\mathscr{F}_1, \cdots, \mathscr{F}_m\}$ joining two vertices v_1 and v_2 in different components K_1 and K_2 of the separated graph. Let $\mathscr{F}_i$ be the last facet of C containing vertices of K_1. If V separates $\mathscr{F}_i$, we are done by Theorem 4. If V does not separate $\mathscr{F}_i$, then $\mathscr{F}_i \cap \mathscr{F}_{i+1}$ contains only separating vertices and by Lemma 3 we are done.

We define an *n-gm* $\mathscr{M}$ to be *simple* provided each k-face belongs to exactly $d-k+1$ facets of $\mathscr{M}$. The proof of the Lower Bound theorem may now be done for simple *n-gm*'s the same way it is done for simple polytopes in [1], using Theorem 13 instead of Sallee's theorem.

We will get the Lower Bound theorem for any triangulated manifold M by constructing a dual M^* of M, that is, a simple *n-gm*. To do so, we shall need two lemmas about triangulated manifolds.

LEMMA 14. *If* $\mathscr{F}$ *is a face of a d-manifold M then* star$(\mathscr{F}, M)$ *is strongly connected.*

PROOF. If star$(\mathscr{F}, M)$ is not strongly connected then it can be separated by a subset of skel_{d-2} star$(\mathscr{F}, M)$. If p is a point in the relative interior of $\mathscr{F}$ then all small neighborhoods of p are separated by skel_{d-2} star$(\mathscr{F}, M)$. But the small neighborhoods of points in M are d-cells and cannot be separated by sets of dimension $d-2$.

LEMMA 15. *If* F_1 *and* F_d *are two k-faces of a d-manifold M (for* $2 \leqq k \leqq d-1$*) that contain a* $(k-1)$*-face H of M then there is a strong sequence of* $(k+1)$*-faces of M containing H and joining* F_1 *and* F_2.

PROOF. The proof follows easily from Lemma 14 and induction on the dimension. We leave it to the reader to supply the details.

If $\mathscr{M}$ is an *n-gm* we define the *semi dual*, $\mathscr{M}^*$ of $\mathscr{M}$ as follows.

(i) Vertices of $\mathscr{M}^*$ correspond to facets of $\mathscr{M}$.

(ii) Two vertices are joined by an edge if and only if the corresponding facets meet on a subfacet.

(iii) Inductively, a set S of l-faces (for $-1 \leqq l \leqq k$) of $\mathscr{M}^*$ form a $(k+1)$-face provided the $(n-l)$-faces of $\mathscr{M}$ corresponding to the faces in S consist of all of the $(n-l)$-faces of $\mathscr{M}$ that contain some $(n-k-1)$-face of $\mathscr{M}$.

With this definition the semi dual of an n-gm may or may not be an n-gm. We can show, however, that if $\mathscr{M}$ is isomorphic to a triangulated manifold then $\mathscr{M}^*$ is a simple n-gm.

THEOREM 16. *If $\mathscr{M}$ is isomorphic to a triangulated manifold M, then $\mathscr{M}^*$ is an n-gm.*

PROOF. First we shall show that each face of $\mathscr{M}^*$ is strongly connected.

Let F be a k-face of $\mathscr{M}^*$ and let H_1 and H_l be two $(k-1)$-faces of F. Corresponding to H_1, H_l and F are $(d-k+1)$-faces $\mathscr{H}_1$ and $\mathscr{H}_l$, and a $(d-k)$-face $\mathscr{F}$ respectively. Let $\mathscr{G}_1, \cdots, \mathscr{G}_l$ be a strong chain of $(d-k+2)$-faces of $\mathscr{M}$ joining $\mathscr{H}_1$ and $\mathscr{H}_l$, and containing $\mathscr{F}$. Corresponding to each $\mathscr{G}_i$ is a $(k-2)$-face G_i of $\mathscr{M}^*$. Corresponding to each $(d-k+1)$-face $\mathscr{G}_i \cap \mathscr{G}_{i+1}$ is a $(k-1)$-face H_i of $\mathscr{M}^*$ that meets H_{i+1} on G_i. This sequence $H_1, \cdots, H_l$ is the desired chain.

Next we show that in any k-face F of $\mathscr{M}^*$, each $(k-2)$-face H belongs to exactly two $(k-1)$-faces. Corresponding to F and H are faces $\mathscr{F}$ and $\mathscr{H}$ in $\mathscr{M}$ of dimensions $(d-k)$ and $(d-k+2)$ respectively, with $\mathscr{F}$ a face of $\mathscr{H}$. In $\mathscr{H}$ there are exactly two $(d-k+1)$-faces, $\mathscr{G}_1$ and $\mathscr{G}_2$, meeting on $\mathscr{F}$. Corresponding to $\mathscr{G}_1$ and $\mathscr{G}_2$ are exactly two faces G_1 and G_2 in $\mathscr{M}^*$ with H belonging to G_1 and G_2.

We now show that the nonempty intersection of any two faces of $\mathscr{M}^*$ is a face of both. Let F^k and F^j be a k-face and a j-face respectively with $F^k \cap F^j \neq \emptyset$. Corresponding to F^k in $\mathscr{M}$ is the set S_1 of all faces containing some $(d-k)$-face $\mathscr{F}^{d-k}$. Corresponding to F^j is the set S_2 of all faces containing a $(d-j)$-face $\mathscr{F}^{d-j}$. Corresponding to $F^k \cap F^j$ is the set of all faces of $\mathscr{M}$ containing both $\mathscr{F}^{d-j}$ and $\mathscr{F}^{d-k}$. If $F^k \cap F^j \neq \emptyset$ then $S_1 \cap S_2 \neq \emptyset$. Thus some face of $\mathscr{M}$ contains $\mathscr{F}^{d-j}$ and $\mathscr{F}^{d-k}$. Let $\mathscr{F}$ be the face of smallest dimension containing $\mathscr{F}^{d-j}$ and $\mathscr{F}^{d-k}$. Then all other faces containing $\mathscr{F}^{d-j}$ and $\mathscr{F}^{d-k}$ will contain $\mathscr{F}$. Thus $F^k \cap F^j$ is the face F of $\mathscr{M}^*$ corresponding to $\mathscr{F}$. We see that F is a face of F^k and of F^k and F^j because $\mathscr{F}$ contains $\mathscr{F}^{d-k}$ and $\mathscr{F}^{d-j}$.

Now we observe that $\mathscr{M}^*$ is simple. If F is a k-face of $\mathscr{M}^*$ there corresponds to it a $(d-k)$-face $\mathscr{F}$ of $\mathscr{M}$. Since all faces of $\mathscr{M}$ are simplices we see that $\mathscr{F}$ contains $d-k+1$ vertices. Thus F belongs to exactly $d-k+1$ facets.

THEOREM 17. *If M is a triangulated d-manifold with v vertices and f facets then* $f \geqq dv - (d+2)(d-1)$.

PROOF. This follows from the Lower Bound theorem for simple n-gm's and from Theorem 16.

REMARK. Theorem 4 was proved independently by D. Walkup (private communication).

REFERENCES

1. D. Barnette, *The minimum number of vertices of a simple d-polytope*, Israel J. Math. **10** (1971), 121–125.

2. D. Barnette and G. Wegner, *A 3-sphere that is not 4-polyhedral*, Studia Sci. Math. Hungar. **6** (1971), 341–346.

3. B. Grünbaum, *Convex Polytopes*, Wiley, New York, 1967.

4. V. Klee, *A property of polyhedral graphs*, J. Math. Mech. **13** (1964), 1039–1042.

5. E. Moise, *Affine structures in 3-manifolds, V (The triangulation theorem and hauptvermutung)*, Ann. of Math. **56** (1952), 96–114.

6. E. Steinitz and H. Radamacher, *Vorlesungen über die Theorie der Polyeder*, Berl'n, 1934.

7. H. Whitney, *Congurent graphs and the connectivity of graphs*, Amer. J. Math. **54** (1932), 150–164.

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